

Massive Hyper-Kähler Sigma Models and BPS Domain Walls

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With the non-Abelian Hyper-Kähler quotient by $U(M)$ and $SU(M)$ gauge groups, we give the massive Hyper-Kähler sigma models that are not toric in the $\mathcal{N} = 1$ superfield formalism. The $U(M)$ quotient gives $N!/ [M!(N-M)!]$ (N is a number of flavors) discrete vacua that may allow various types of domain walls, whereas the $SU(M)$ quotient gives no discrete vacua. We derive BPS domain wall solution in the case of $N = 2$ and $M = 1$ in the $U(M)$ quotient model.

I. INTRODUCTION

It is well known that topological solutions are of importance in various areas of particle physics. Recently, there was renewed interest in such solutions because of their crucial role in the brane world scenario [1,2]. In this brane-world scenario, our four-dimensional world is to be realized on topological objects like domain walls or brane-junctions. Supersymmetry (SUSY) can be implemented in these models, and it is actually a powerful device for constructing their topological solutions. Viewing the four-dimensional world as a domain wall, we are led to deal with SUSY theories with eight supercharges in five dimensions.

SUSY with eight supercharges is very restrictive. In theories involving only massless scalar multiplets (hypermultiplets), non-trivial interactions can only arise from nonlinearities in kinetic term, say nonlinear sigma models (NLSMs). Prior to studying the genuine five-dimensional theories with hypermultiplets, it is instructive to start with similar SUSY theories in four dimensions, i.e., $\mathcal{N} = 2$, $d = 4$ theories. The analysis of the four-dimensional

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theory could then be of help in studying the brane world scenario based on SUSY theories in higher dimensions [3].

With regard to rigid $\mathcal{N} = 2$ SUSY the target manifold of the hypermultiplet $d = 4$ NLSMs must be Hyper-Kähler (HK) [4]. In these theories, the scalar potential can be obtained only if the hypermultiplets acquire masses by the Scherk-Schwarz mechanism [5] because of the appearance of central charges in the $\mathcal{N} = 2$ Poincaré superalgebra [6]. The NLSMs with the scalar potential in $\mathcal{N} = 2$ theories are called the massive HK NLSMs.

A large class of HK manifold is given by toric HK manifolds that are defined as HK manifolds of real dimension $4n$ admitting mutually commuting n Abelian tri-holomorphic isometries. In the massive HK NLSMs on toric HK manifolds, many interesting BPS solitons were constructed in the component formalism [7–10] as well as off-shell formulation [11–13]. The potential term of the massive $T^*\mathbf{CP}^{N-1}$ model which is toric comes from the mass terms of the hypermultiplets when the NLSM is constructed as the quotient by the $U(1)$ gauge group [11,12]. We call this formulation of massive HK NLSMs as “the massive HK quotient method”, since massless case is just a HK quotient found in Refs. [14,15]. One of the advantages of our massive HK quotient is that the off-shell formulation of the SUSY NLSMs is possible [12]. Off-shell formulation is powerful to extend the models to those with other isometries, and/or gauge symmetries and to those coupled with gravity, since (part of) SUSY is manifest. Any *toric* HK manifolds can be constructed using an *Abelian* HK quotient [16,17]. Therefore an off-shell formulation of general massive toric HK NLSMs [8] can be obtained using the massive HK quotient with the Abelian gauge theories. On the other hand, a massless HK NLSM other than the toric HK target manifolds has been obtained as a quotient using non-Abelian gauge group by Lindström and Roček [14] for *massless* case only (without potential terms).

In this talk, we discuss massive NLSMs in $\mathcal{N} = 2$, $d = 4$ theories and their BPS domain wall solutions. With HK quotient method, massive NLSMs on cotangent bundles over the Grassmann manifolds, $T^*G_{N,M}$, which are not toric, are obtained along with their generalization. These models are constructed in $\mathcal{N} = 1$ superfield formalism. BPS domain wall solution is given in the simplest case, the Eguchi-Hanson target manifold [18] ($N = 2$ and $M = 1$). This talk is based on our papers [12,19] in which analysis by a fully off-shell $\mathcal{N} = 2$ superspace (the harmonic superspace [20]) formalism is also discussed in detail.

II. MASSIVE HK QUOTIENT BY $U(M)$ GAUGE GROUP

We consider $\mathcal{N} = 2$ SUSY QCD with N -flavors and a $U(M)$ gauge group. In terms of $\mathcal{N} = 1$ superfields, $\mathcal{N} = 2$, NM hypermultiplets can be decomposed into $(N \times M)$ - and $(M \times N)$ -matrix chiral superfields $\Phi(x, \theta, \bar{\theta})$ and $\Psi(x, \theta, \bar{\theta})$, and $\mathcal{N} = 2$ vector multiplets for the $U(M)$ gauge symmetry can be decomposed into $M \times M$ matrices of $\mathcal{N} = 1$ vector

superfields $V = V^A(x, \theta, \bar{\theta})T_A$ and chiral superfields $\Sigma = \Sigma^A(x, \theta, \bar{\theta})T_A$, with $M \times M$ matrices T_A ($A = 1, \dots, M$) of the fundamental representation of the generators of the $U(M)$ gauge group. In order that the vector multiplets are treated as Lagrange multipliers, we take strong coupling limit $g \rightarrow \infty$, and drop the kinetic term. The gauge invariant Lagrangian is given by

$$\begin{aligned} \mathcal{L} = & \int d^4\theta \left[\text{tr}(\Phi^\dagger \Phi e^V) + \text{tr}(\Psi \Psi^\dagger e^{-V}) - c \text{tr} V \right] \\ & + \left[\int d^2\theta \left(\text{tr} \{ \Sigma (\Psi \Phi - b \mathbf{1}_M) \} + \sum_{a=1}^{N-1} m_a \text{tr}(\Psi H_a \Phi) \right) + \text{c.c.} \right], \end{aligned} \quad (1)$$

where we have absorbed a common mass of hypermultiplets into the field Σ , denoted m_a ($a = 1, \dots, N-1$) as complex mass parameters and H_a are diagonal traceless matrices, interpreted as the Cartan generators of $SU(N)$ below. The electric and magnetic Fayet-Iliopoulos (FI) parameters are denoted as $c \in \mathbf{R}$ and $b \in \mathbf{C}$, respectively. Note that $U(M)$ gauge symmetry is complexified.

Next we eliminate the auxiliary superfields V and Σ in the superfield formalism. Their equations of motion read from Eq.(1):

$$\frac{\partial \mathcal{L}}{\partial V} = \Phi^\dagger \Phi e^V - e^{-V} \Psi \Psi^\dagger - c \mathbf{1}_M = 0, \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \Sigma} = \Psi \Phi - b \mathbf{1}_M = 0. \quad (3)$$

From the first equation, V can be solved

$$e^V = \frac{c}{2} (\Phi^\dagger \Phi)^{-1} \left(\mathbf{1}_M \pm \sqrt{\mathbf{1}_M + \frac{4}{c^2} \Phi^\dagger \Phi \Psi \Psi^\dagger} \right). \quad (4)$$

Substituting this back into (1), we obtain the Kähler potential for the Lindström-Roček metric [14]

$$K = c \text{tr} \sqrt{\mathbf{1}_M + \frac{4}{c^2} \Phi^\dagger \Phi \Psi \Psi^\dagger} - c \text{tr} \log \left(\mathbf{1}_M + \sqrt{\mathbf{1}_M + \frac{4}{c^2} \Phi^\dagger \Phi \Psi \Psi^\dagger} \right) + c \text{tr} \log \Phi^\dagger \Phi. \quad (5)$$

Fixing the complexified $U(M)$ gauge symmetry and solving constraint (3), we obtain the Lagrangian of the NLSM in terms of independent superfields. To this end we should consider two cases i) $b = 0$ and ii) $b \neq 0$ separately.

i) $b = 0$. In this case, a gauge can be fixed as

$$\Phi = \begin{pmatrix} \mathbf{1}_M \\ \varphi \end{pmatrix}, \quad \Psi = (-\psi \varphi, \psi), \quad (6)$$

with φ and ψ being $[(N-M) \times M]$ - and $[M \times (N-M)]$ -matrix chiral superfields, respectively. The superpotential becomes

$$W = \sum_a m_a \text{tr} \left[(-\psi\varphi, \psi) H_a \begin{pmatrix} \mathbf{1}_M \\ \varphi \end{pmatrix} \right] = \sum_a m_a \text{tr} \left[H_a \begin{pmatrix} -\psi\varphi & \psi \\ -\varphi\psi\varphi & \varphi\psi \end{pmatrix} \right]. \quad (7)$$

ii) $b \neq 0$. In this case, we can take a gauge as [14]

$$\Phi = \begin{pmatrix} \mathbf{1}_M \\ \varphi \end{pmatrix} Q, \quad \Psi = Q(\mathbf{1}_M, \psi), \quad Q = \sqrt{b}(\mathbf{1}_M + \psi\varphi)^{-\frac{1}{2}}, \quad (8)$$

with φ and ψ being again $[(N - M) \times M]$ - and $[M \times (N - M)]$ -matrix chiral superfields, respectively. In this case, the superpotential is given by

$$W = b \sum_a m_a \text{tr} \left[H_a \begin{pmatrix} \mathbf{1}_M \\ \varphi \end{pmatrix} (\mathbf{1}_M + \psi\varphi)^{-1} (\mathbf{1}_M, \psi) \right]. \quad (9)$$

These two cases are not holomorphically transformed to each other, because they make different complex structures manifest.

We can find the bundle structure of the manifold as follows: i) $b = 0$. Putting $\psi = 0$, the Kähler potential becomes

$$K|_{\psi=0} = c \text{tr} \log(1 + \varphi^\dagger \varphi), \quad (10)$$

which is the one of the Grassmann manifold. Therefore φ parameterize the base Grassmann manifold, whereas ψ the cotangent space as the fiber, with the total space being the cotangent bundle over the Grassmann manifold $T^*G_{N,M}$. ii) $b \neq 0$. In the case of $T^*\mathbf{CP}^{N-1}$ of $M = 1$, the base manifold is embedded by $\varphi = \psi^\dagger$ [21].¹

There exists the manifest duality between two theories with $U(M)$ gauge and $U(N - M)$ gauge symmetries and the same flavor $SU(N)$ symmetry. This comes directly from the duality in the base Grassmann manifold $G_{N,M} \simeq G_{N,N-M}$.

For $M = 1$ ($M = N - 1$) namely for the $U(1)$ [$U(N - 1)$] gauge symmetry, this model reduces to $T^*\mathbf{CP}^{N-1} \simeq T^*G_{N,1} (\simeq T^*G_{N,N-1})$ [22] which we discussed in detail in [12]. Moreover if $N = 2$ the manifold $T^*\mathbf{CP}^1$ is the Eguchi-Hanson space. A nontrivial model in the lowest dimensions other than $T^*\mathbf{CP}^{N-1}$ is the case of $N = 4, M = 2$. The manifold is $T^*G_{4,2} = T^*[SU(4)/SU(2) \times SU(2) \times U(1)] = T^*[SO(6)/SO(4) \times U(1)] \equiv T^*Q^4$ in which the base manifold Q^4 is called the Klein quadric space.

¹This embedding $\varphi = \psi^\dagger$ should hold for a matrix of general M , although we have not proved it yet.

III. VACUUM STRUCTURE

A. Vacua in the massive $T^*\mathbf{CP}^{N-1}$ model

In this subsection we discuss $T^*\mathbf{CP}^{N-1} = T^*G_{N,1}$ of $M = 1$. Without loss of generality we consider the case of $b = 0$ and $c \neq 0$. The dynamical matrix fields are column and row vectors like $\varphi^T = (\varphi^1, \dots, \varphi^{N-1})$ and $\psi = (\psi^1, \dots, \psi^{N-1})$.

The superpotential given in (7) becomes

$$W = \sum_a m_a \text{tr} \left[H_a \begin{pmatrix} -\psi \cdot \varphi & \psi \\ -\varphi(\psi \cdot \varphi) & \varphi \otimes \psi \end{pmatrix} \right]. \quad (11)$$

We take H_a ($a = 1, \dots, N-1$) as

$$H_a = \frac{1}{\sqrt{a(a+1)}} \text{diag.} (1, \dots, 1, -a, 0, \dots, 0), \quad (12)$$

where $-a$ is the $(a+1)$ -th component, with a normalization given by the trace $\text{tr}(H_a H_b) = \delta_{ab}$. Then the superpotential can be calculated as

$$W = - \sum_a M_a \psi^a \varphi^a, \quad M_a \equiv \sqrt{\frac{a}{a+1}} m_a + \sum_{b=1}^a \frac{m_b}{\sqrt{b(b+1)}}. \quad (13)$$

Therefore the derivatives of W with respect to fields are

$$\partial_{\varphi^a} W = -M_a \psi^a, \quad \partial_{\psi^a} W = -M_a \varphi^a \quad (\text{no sum}). \quad (14)$$

These vanish only at the origin $\varphi = \psi^T = 0$, which is the only one vacuum in the regular region of these coordinates because the metric is regular there.

This model, however, contains more vacua, because the whole manifold is covered by the several coordinate patches and the vacuum exists at the origin of each coordinate patch. To see this we concentrate on the base \mathbf{CP}^{N-1} for a while. We consider the fields before the gauge fixing, $\Phi \equiv \phi^A = (\phi^1, \dots, \phi^N)^T$ ($A = 1, \dots, N$) called the homogeneous coordinates, in which we need an identification by the gauge transformation $\phi^A \sim e^{i\Lambda} \phi^A$. In the region $\phi^1 \neq 0$ we can take a patch $\varphi^i = \phi^{i+1}/\phi^1$ ($i = 1, \dots, N-1$), which was used in Eq. (6). Here let us write these coordinates as $\varphi_{(1)}^i = \phi^{i+1}/\phi^1$. In the same way, in the region of $\phi^A \neq 0$, we can take the A -th patch defined by

$$\varphi_{(A)}^i = \begin{cases} \phi^i/\phi^A & (1 \leq i \leq A-1) \\ \phi^{i+1}/\phi^A & (A \leq i \leq N-1) \end{cases}. \quad (15)$$

We thus have N sets of patches $\{\varphi_{(A)}^i\}$ enough to cover the whole base manifold. Corresponding to each patch for the base space, we manifestly have an associated patch for the

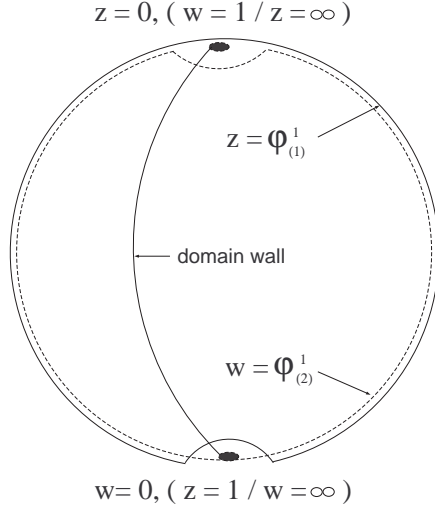


FIG. 1. The base manifold of $T^*\mathbf{CP}^1$ and vacua.

Corresponding to two gauge fixing conditions, we have two coordinates z and w , covering S^2 except for South (S) and North (N) Poles, respectively. The origins of z and w (N and S, respectively) are both vacua. The domain wall solution, approaching to these two vacua in spatial infinities, is mapped to a trajectory connecting N and S in S^2 .

fiber tangent space $\{\psi_{(A)}^i\}$ from Eq. (6). These sets of coordinates $\{\varphi_{(A)}^i, \psi_{(A)}^i\}$ are enough to cover the whole $T^*\mathbf{CP}^{N-1}$. For each patch, the origin $\varphi_{(A)}^i = \psi_{(A)}^i = 0$ is a vacuum. Therefore the number of discrete vacua for the massive $T^*\mathbf{CP}^{N-1}$ model is N , which was firstly found in [9].

To discuss solitons like BPS walls, their junction and lumps, it may be better to consider the problem in one coordinate patch. The other vacua appear in one patch as the coordinate singularities of the metric in infinities of the coordinates rather than the stationary points of the superpotential [23]. To see this, we consider only the base \mathbf{CP}^{N-1} once again. We discuss how the A -th vacuum ($A \neq 1$) in the origin of the A -th coordinate patch is mapped in the first patch. The A -th vacuum is represented by $\varphi_{(A)}^i = 0$ or $\phi^B/\phi^A = 0$ for all $B(\neq A)$. In the first coordinate patch, this point is mapped into an infinite point represented by

$$\varphi_{(1)}^{A-1} \rightarrow \infty, \quad \varphi_{(1)}^i/\varphi_{(1)}^{A-1} \rightarrow 0 \quad (i \neq A-1), \quad (16)$$

which looks like a runaway vacuum in this patch. Hence, the origin and $N-1$ infinities are vacua in each coordinate patch [23]. As a summary, if we include runaway vacua, one patch is enough to describe soliton solutions. However note that the terminology “runaway” is just a coordinate-dependent concept, because a runaway vacuum in one coordinate patch is a true vacuum in the other coordinate patch.

We can also discuss the vacua without referring to the local coordinate patches. We

concentrate on the base \mathbf{CP}^{N-1} once again. A point in the \mathbf{CP}^{N-1} corresponds to a complex line through the origin in \mathbf{C}^N with homogeneous coordinates ϕ^A , because of the gauge transformation $\phi^A \sim e^{i\Lambda} \phi^A$ as an equivalence relation. The first vacuum is expressed in region $\phi^1 \neq 0$ by $\varphi_{(1)}^i = \phi^{i+1}/\phi^1 = 0$ ($i = 1, \dots, N-1$), namely $\phi^{i+1} = 0$. Therefore the first vacuum corresponds to the ϕ^1 -axis. In the same way, the A -th vacuum corresponds to the ϕ^A -axis. Each vacuum is simply expressed by each orthogonal axis in \mathbf{C}^N . Note that each axis is invariant under $U(1)^{N-1}$ transformation of H_a so that it is a fixed point of this transformation.

If we take N orthogonal normalized basis e_A [with $(e_A)^* \cdot e_B = \delta_{AB}$] whose components are given by

$$(e_A)^B = \delta_A^B, \quad (17)$$

a complex line in \mathbf{C}^N can be spanned by an unit vector $e' = \sum_{A=1}^N a^A e_A = U e_1$ where a^A is a complex number with $\sum_A |a^A|^2 = 1$ and U is a unitary matrix $U \in U(N)$. Each of the N -vacua found above corresponds to each e_A ($A = 1, \dots, N$) (with zero value of the cotangent space $\psi = 0$).

Example: the Eguchi-Hanson space [18]. The simplest model is the Eguchi-Hanson space, $T^*\mathbf{CP}^1$ ($N = 2$ and $M = 1$). This model has two discrete vacua and admits a typical domain wall solution [7,12]. The vacua are located on the North and South Poles of the base $\mathbf{CP}^1 \simeq S^2$ (see Fig. 1). Corresponding to two gauge fixing conditions $\Phi = \begin{pmatrix} 1 \\ z \end{pmatrix}$ and $\Phi = \begin{pmatrix} w \\ 1 \end{pmatrix}$, we have two coordinate patches $z \equiv \varphi_{(1)}^1 = \phi^2/\phi^1$ and $w \equiv \varphi_{(2)}^1 = \phi^1/\phi^2$, which are related by $z = 1/w$. Two vacua are given by $z = 0$ and $w = 0$. The second (first) vacuum $w = 0$ ($z = 0$) is mapped to $z = \infty$ ($w = \infty$) in the first (second) patch, which looks like a runaway vacuum. In homogeneous coordinates, these correspond to $\langle \Phi \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv e_1$ and $\langle \Phi \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv e_2$, respectively, with $\langle \Psi \rangle = (0, 0)$. Also, in the coordinate independent way, these two vacua correspond to the ϕ^1 and ϕ^2 axes spanned by e_1 and e_2 , respectively.

Before closing this subsection, we discuss the case of $b \neq 0$. The superpotential (9) can be calculated, to give

$$W = \frac{b}{1 + \psi \cdot \varphi} \left(L + \sum_{a=1}^{N-1} N_a \psi^a \varphi^a \right),$$

$$L \equiv \sum_{a=1}^{N-1} \frac{m_a}{\sqrt{a(a+1)}}, \quad N_a \equiv -\sqrt{\frac{a}{a+1}} m_a + \sum_{b=a+1}^{N-1} \frac{m_b}{\sqrt{b(b+1)}} = L - M_a, \quad (18)$$

with M_a defined in (13). The derivatives of W are

$$\begin{aligned}\partial_{\varphi^a} W &= -\frac{b\psi^a}{(1+\psi\cdot\varphi)^2} \left[M_a - \sum_{b=1}^{N-1} (M_b - M_a)\psi^b\varphi^b \right], \\ \partial_{\psi^a} W &= (\psi^a \leftrightarrow \varphi^a),\end{aligned}\tag{19}$$

where an arrow in the second equation represents the exchange of quantities in the first equation. The origin $\varphi^a = \psi^a = 0$ in each patch is a vacuum. There is no other vacuum than these N vacua; The number of vacua should coincide with the case of $b = 0$ and $c \neq 0$, because they are connected by the R-symmetry and the physics does not depend on the difference.

B. Vacua in the massive $T^*G_{N,M}$ model

To look for vacua of the $T^*G_{N,M}$ model, we consider the case $b = 0$ and $c \neq 0$ again without loss of generality. We label the indices for the matrices as $\varphi = (\varphi_{i\alpha})$ and $\psi = (\psi_{\alpha i})$ in which $i = 1, \dots, N - M$ and $\alpha = 1, \dots, M$. The superpotential given in Eq. (7) can be calculated as

$$\begin{aligned}W &= -\sum_{\alpha=1}^M \sum_{i=1}^{N-M} M_{\alpha i} \varphi_{i\alpha} \psi_{\alpha i}, \\ M_{\alpha i} &\equiv \sqrt{\frac{i+M-1}{i+M}} m_{i+M-1} - \sqrt{\frac{\alpha-1}{\alpha}} m_{\alpha-1} + \sum_{a=\alpha}^{i+M-1} \frac{m_a}{\sqrt{a(a+1)}},\end{aligned}\tag{20}$$

where we have set $m_0 \equiv 0$. For the case of $M = 1$ ($\alpha = 1$), this reduces to Eq. (13) for $T^*\mathbf{CP}^{N-1}$. From the superpotential (20), its derivatives with respect to the fields are

$$\partial_{\varphi_{i\alpha}} W = -M_{\alpha i} \psi_{\alpha i}, \quad \partial_{\psi_{\alpha i}} W = -M_{\alpha i} \varphi_{i\alpha} \quad (\text{no sum}).\tag{21}$$

Therefore the origin of these coordinates, $\varphi = \psi^T = 0$, is a vacuum, and this is the only one vacuum in the finite region of these coordinates where the metric is regular. This model contains vacua as many as the coordinate patches, like the $T^*\mathbf{CP}^{N-1}$ case. In the first coordinate patch, we have chosen the first M row vectors in Φ the unit matrix as in Eqs. (6) or (8). The other coordinate patches are given by the other choices of gauge fixing conditions making the other sets of M row vectors in Φ the unit matrix. The number of such coordinate systems is ${}_N C_M = N!/[M!(N-M)!]$. They are independent and enough to cover the whole manifold, so this model has $N!/[M!(N-M)!]$ vacua. This number is invariant under the duality between $U(M)$ and $U(N-M)$ gauge groups. It also reduces correctly to N for $T^*\mathbf{CP}^{N-1}$ when $M = 1$ or $M = N - 1$.

As in the $T^*\mathbf{CP}^{N-1}$ case, we can understand the vacua of $T^*G_{N,M}$ without local coordinates. A point in the base $G_{N,M}$ corresponds to an M -dimensional complex plane through the origin in \mathbf{C}^N . The vacua found above correspond to mutually orthogonal M -planes

spanned by arbitrary M sets of axes chosen from the N axes. Therefore the total number of vacua is ${}_NC_M = N!/[M!(N-M)!]$. Since the M -planes of vacua are invariant under $U(1)^{N-1}$ generated by H_a , the vacua are fixed points.

Taking basis (17) in \mathbf{C}^N , a point in $G_{N,M}$ expressed by an M -plane in \mathbf{C}^N can be spanned by M set of unit vectors

$$(e_i)' = Ue_i, \quad (22)$$

where $i = 1, \dots, N-M$ and U is an unitary matrix, $U \in U(N)$. The vacua of mutually orthogonal M -planes are spanned by arbitrary M sets of basis among orthogonal N -basis,

The duality becomes manifest in this framework. We can represent a point in $G_{N,M}$ by an $(N-M)$ -plane complement to an M -plane.

Example: the cotangent bundle over the Klein quadric. An example is given for the Klein quadric $T^*G_{4,2} = T^*Q^4$ ($N = 4$ and $M = 2$). There exist six coordinate systems $\varphi_{i\alpha}^{(A)}$ ($A = 1, \dots, 6$) for the base manifold corresponding to six choices of gauge fixing, given by

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \varphi_{11}^{(1)} & \varphi_{12}^{(1)} \\ \varphi_{21}^{(1)} & \varphi_{22}^{(1)} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \varphi_{11}^{(2)} & \varphi_{12}^{(2)} \\ 0 & 1 \\ \varphi_{21}^{(2)} & \varphi_{22}^{(2)} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \varphi_{11}^{(3)} & \varphi_{12}^{(3)} \\ \varphi_{21}^{(3)} & \varphi_{22}^{(3)} \\ 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} \varphi_{11}^{(4)} & \varphi_{12}^{(4)} \\ 1 & 0 \\ 0 & 1 \\ \varphi_{21}^{(4)} & \varphi_{22}^{(4)} \end{pmatrix}, \begin{pmatrix} \varphi_{11}^{(5)} & \varphi_{12}^{(5)} \\ 1 & 0 \\ \varphi_{21}^{(5)} & \varphi_{22}^{(5)} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \varphi_{11}^{(6)} & \varphi_{12}^{(6)} \\ \varphi_{21}^{(6)} & \varphi_{22}^{(6)} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (23)$$

Together with corresponding coordinates $\psi_{\alpha i}^{(A)}$ for the cotangent space in Eq. (6), these six sets of coordinate systems are enough to cover the whole manifold. Therefore this model has the six vacua given by

$$\langle \Phi \rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (24)$$

which are the origins of (23) respectively, with $\langle \Psi \rangle = 0$. A set of two column vectors in each matrix in Eq. (24) is a set of orthogonal basis e_i chosen from the four basis.

In the case of $b \neq 0$, the superpotential (9) is

$$\begin{aligned}
W &= b \sum_{a=1}^{N-1} \sum_{n=0}^{\infty} (-1)^n m_a \text{tr} \left[H_a \begin{pmatrix} (\psi\varphi)^n & (\psi\varphi)^n \psi \\ \varphi(\psi\varphi)^n & (\varphi\psi)^{n+1} \end{pmatrix} \right] \\
&= b \sum_{a=1}^{N-1} \sum_{n=0}^{\infty} (-1)^n m_a \text{tr} \left[H_a \begin{pmatrix} (\psi\varphi)^n & 0 \\ 0 & (\varphi\psi)^{n+1} \end{pmatrix} \right], \tag{25}
\end{aligned}$$

where the last equality holds because H_a are diagonal. Similarly to the $T^*\mathbf{CP}^{N-1}$ case, the origin $\varphi = \psi^T = 0$ of each patch is a vacuum and we cannot have any other vacua.

IV. MASSIVE HK QUOTIENT BY $SU(M)$ GAUGE GROUP

In this section, we construct the massive HK NLSM with the $SU(M)$ gauge group. We eliminate the vector multiplets in the superfield formalism and find that this model does not have discrete vacua.

A. Massive HK NLSM by SU gauge group

In this subsection, we consider $\mathcal{N} = 2$ SUSY QCD with N -flavors and the $SU(M)$ gauge group. We take the same matter field contents with $T^*G_{N,M}$ but gauge multiplets take values in the Lie algebra of $SU(M)$: $V = V^A T_A$ and $\Sigma = \Sigma^A T_A$ with T_A generators of $SU(M)$. Then the Lagrangian is given by

$$\begin{aligned}
\mathcal{L} &= \int d^4\theta \left[\text{tr}(\Phi^\dagger \Phi e^V) + \text{tr}(\Psi \Psi^\dagger e^{-V}) \right] \\
&\quad + \left[\int d^2\theta \left(\text{tr}(\Sigma \Psi \Phi) + \sum_{a=1}^{N-1} m_a \text{tr}(\Psi H_a \Phi) \right) + \text{c.c.} \right]. \tag{26}
\end{aligned}$$

We do not have any FI parameters because of the absence of any $U(1)$ gauge symmetry. The $SU(M)$ gauge transformation is given by the same way as in the $U(M)$ case and it is complexified to $SU(M)^\mathbf{C} = SL(M, \mathbf{C})$. This model has an additional $U(1)_\text{D}$ flavor symmetry,

$$\Phi \rightarrow \Phi' = e^{i\lambda} \Phi, \quad \Psi \rightarrow \Psi' = e^{-i\lambda} \Psi, \tag{27}$$

which was gauged in $U(M)$ case.

We eliminate all auxiliary superfields in the superfield formalism. Equations of motion for V, Σ imply

$$\Phi^\dagger \Phi e^V - e^{-V} \Psi \Psi^\dagger = C \mathbf{1}_M, \tag{28}$$

$$\Psi \Phi = B \mathbf{1}_M, \tag{29}$$

respectively, with $C(x, \theta, \bar{\theta})$ and $B(x, \theta, \bar{\theta})$ being vector and chiral superfields in the $\mathcal{N} = 1$ superfields formalism.

The gauge field V can be solved in terms of the dynamical fields from Eq. (28) as

$$e^V = \frac{1}{2}(\Phi^\dagger \Phi)^{-1} \left(C \mathbf{1}_M \pm \sqrt{C^2 \mathbf{1}_M + 4\Phi^\dagger \Phi \Psi \Psi^\dagger} \right). \quad (30)$$

Since the equation $\det e^V = 1$ holds, we get the equation

$$\det \left(C \mathbf{1}_M \pm \sqrt{C^2 \mathbf{1}_M + 4\Phi^\dagger \Phi \Psi \Psi^\dagger} \right) = 2^M \det(\Phi^\dagger \Phi) \quad (31)$$

which enables us to express C in terms of dynamical fields implicitly: $C = C(\Phi, \Phi^\dagger; \Psi, \Psi^\dagger)$. On the other hand, Eq. (29) implies

$$B = \frac{1}{M} \text{tr}(\Phi \Psi). \quad (32)$$

Substituting the solution (30) back into the Lagrangian (26), we obtain the Kähler potential

$$K = \pm \text{tr} \sqrt{C^2(\Phi, \Phi^\dagger; \Psi, \Psi^\dagger) \mathbf{1}_M + 4\Phi^\dagger \Phi \Psi \Psi^\dagger}, \quad (33)$$

with C satisfying the constraint (31). We should choose the plus sign for the positivity of the metric.

Let us fix the complex gauge symmetry $SU(M)^{\mathbf{C}} = SL(M, \mathbf{C})$ to express the Lagrangian in terms of independent superfields. We can take the similar gauge as the $b \neq 0$ case in $T^*G_{N,M}$:

$$\Phi = \sigma \begin{pmatrix} \mathbf{1}_M \\ \varphi \end{pmatrix} P, \quad \Psi = P(\mathbf{1}_M, \psi) \rho, \quad P = (\mathbf{1}_M + \psi \varphi)^{-\frac{1}{2}}, \quad (34)$$

with φ and ψ being $[(N-M) \times M]$ - and $[M \times (N-M)]$ -matrix chiral superfields, respectively. Here, σ and ρ are chiral superfields satisfying $\sigma \rho = B$ from Eq. (32). We can consider σ and ρ independent fields among these three fields.

Substituting Eq. (34) into the Kähler potential (33), we obtain the Kähler potential in terms of independent fields $\varphi, \psi, \rho, \sigma$ and their conjugates. The superpotential also can be calculated as

$$W = \sum_a m_a \sigma \rho \text{tr} \left[H_a \begin{pmatrix} \mathbf{1}_M \\ \varphi \end{pmatrix} (\mathbf{1}_M + \psi \varphi)^{-1} (\mathbf{1}_M, \psi) \right]. \quad (35)$$

This target manifold has the isometry of $U(N) = SU(N) \times U(1)_D$, in which the $SU(N)$ part is the same with $T^*G_{N,M}$. The Kähler potential does not receive the Kähler transformation. As for the symmetry of the Lagrangian, the superpotential is invariant under the $U(1)$ fiber symmetry originated from (27)

$$\sigma \rightarrow \sigma' = e^{i\lambda} \sigma, \quad \rho \rightarrow \rho' = e^{-i\lambda} \rho, \quad (36)$$

besides the $U(1)^{N-1}$ symmetry of the massive $T^*G_{N,M}$ model. Gauging this $U(1)_D$ symmetry, we obtain the $T^*G_{N,M}$ model. Gauging $U(1)_D$ symmetry implies putting B and C in the constraints (28) and (29) as constants and the constraints then become $T^*G_{N,M}$'s ones (2) and (3), respectively. This clarifies the bundle structure: the set of σ and ρ is a fiber of quaternion with the total manifold being the (quaternionic) line bundle over $T^*G_{N,M}$.

B. Vacua of SU gauge theories

We look for the vacua of the HK NLSM by the SU gauge group. The superpotential (35) of this model can be rewritten as

$$W = \sigma \rho \sum_{a=1}^{N-1} \sum_{n=0}^{\infty} (-1)^n m_a \text{tr} \left[H_a \begin{pmatrix} (\psi\varphi)^n & 0 \\ 0 & (\varphi\psi)^{n+1} \end{pmatrix} \right] \equiv \sigma \rho W_U, \quad (37)$$

where W_U (times b) denotes the superpotential (9) or (25) of the $U(M)$ gauge group with $b \neq 0$. The derivatives of the superpotential with respect to fields are given by $\partial_\psi W = \sigma \rho \partial_\psi W_U$, $\partial_\varphi W = \sigma \rho \partial_\varphi W_U$, $\partial_\rho W = \sigma W_U$ and $\partial_\sigma W = \rho W_U$. The vacuum condition is given by $\sigma = \rho = 0$, since $\partial W_U = 0$ holds only at $\varphi = \psi^T = 0$ from the discussion in the last section, but $W_U \neq 0$ there. Therefore this model has no discrete vacua, and so we cannot expect any wall solutions.

V. BPS EQUATION AND ITS SOLUTION

In this section, we construct the BPS domain wall in $N = 2$ and $M = 1$ case of $T^*G_{N,M}$ i.e., $T^*\mathbb{CP}^1$. In what follows, we consider $b \neq 0$ and $c = 0$ case. We assume that there exists domain wall solution perpendicular to $y = x^2$ direction. BPS domain wall solution is derived from vanishing of the SUSY transformation for fermions

$$0 = i\sqrt{2}\sigma^\mu \bar{\epsilon} \partial_\mu \Phi^i + \sqrt{2}\epsilon F^i \quad (38)$$

with half SUSY condition $e^{i\alpha}\sigma^2\bar{\epsilon} = i\epsilon$ where $e^{i\alpha}$ is a phase factor, Φ^i and F^i are scalar and auxiliary fields, respectively. In the case we consider now, the scalar field is given by $\Phi^i = \sqrt{\frac{b}{1+\varphi\psi}} \begin{pmatrix} 1 \\ \varphi \end{pmatrix}$ from Eq. (8). Eliminating the auxiliary fields, the BPS equations are given by

$$\partial_2 \varphi^i = -e^{i\alpha} g^{ij*} \partial_{j*} W^*, \quad (39)$$

where g^{ij*} is inverse of the metric $g_{ij*} = \partial_i \partial_{j*} K$ and K is given by (5) with (8). Substituting the metric and the superpotential (9), these BPS equations reduce to

$$\begin{aligned}
\partial_2 \varphi &= e^{i\alpha} \frac{m^*}{4b} K(1 + \varphi\psi)^2 \left[\frac{|1 + \varphi\psi|^2 + (1 + |\varphi|^2)(1 + |\psi|^2)}{|1 + \varphi\psi|^2(1 + |\psi|^2)^2} \psi^* + \frac{(\varphi - \psi^*)^2 \varphi^*}{|1 + \varphi\psi|^2(1 + |\varphi|^2)(1 + |\psi|^2)} \right], \\
\partial_2 \psi &= e^{i\alpha} \frac{m^*}{4b} K(1 + \varphi\psi)^2 \left[\frac{|1 + \varphi\psi|^2 + (1 + |\varphi|^2)(1 + |\psi|^2)}{|1 + \varphi\psi|^2(1 + |\varphi|^2)^2} \varphi^* + \frac{(\psi - \varphi^*)^2 \psi^*}{|1 + \varphi\psi|^2(1 + |\varphi|^2)(1 + |\psi|^2)} \right],
\end{aligned} \tag{40}$$

where m is a mass parameter. Now we must choose the phase $e^{i\alpha}$ to absorb the phase of the parameter m^*/b

$$e^{i\alpha} \frac{m^*}{b} = \left| \frac{m}{b} \right|. \tag{41}$$

By subtracting the complex conjugate of the second equation from the first one in Eq.(40), we obtain

$$\begin{aligned}
\frac{\partial(\varphi - \psi^*)}{\partial y} &= \left| \frac{m}{b} \right| \frac{K}{4} \left[\left\{ \left(\frac{1 + \varphi\psi}{|1 + \varphi\psi|} \right)^2 \varphi^* - \left(\frac{1 + \varphi^* \psi^*}{|1 + \varphi\psi|} \right)^2 \psi \right\} \frac{(\varphi - \psi^*)^2}{(1 + |\varphi|^2)(1 + |\psi|^2)} \right. \\
&\quad \left. + \left\{ \left(\frac{1 + \varphi\psi}{|1 + \varphi\psi|} \right)^2 \frac{\psi^*}{(1 + |\psi|^2)^2} - \left(\frac{1 + \varphi^* \psi^*}{|1 + \varphi\psi|} \right)^2 \frac{\varphi}{(1 + |\varphi|^2)^2} \right\} \{ |1 + \varphi\psi|^2 + (1 + |\varphi|^2)(1 + |\psi|^2) \} \right],
\end{aligned} \tag{42}$$

whose right-hand side vanishes for $\varphi = \psi^*$. The BPS equation (42) dictates that $\varphi = \psi^*$ is valid for arbitrary y , if an initial condition $\varphi = \psi^*$ is chosen at some y . Since we can choose the initial condition $\varphi = \psi^*$ at $y = -\infty$, we find the BPS equations (40) simply reduce to

$$\partial_2 \varphi = |m| \varphi, \tag{43}$$

which is the BPS equation on the submanifold \mathbf{CP}^1 defined by $\varphi = \psi^*$ [12]. Therefore we obtain a BPS wall configuration connecting two vacua $\varphi = \psi^* = 0$ at $y = -\infty$ to $\varphi = \psi^* = \infty$ at $y = \infty$ along $\varphi = \psi^*$ with a constant phase $e^{i\phi_0}$

$$\varphi = \psi^* = e^{|m|(y+y_0)} e^{i\phi_0}, \tag{44}$$

where y_0 is also a constant representing the position of the wall. Thus we find two collective coordinates (zero modes) corresponding to the spontaneously broken translation (y_0) and $U(1)$ symmetry (ϕ_0).

We can show that BPS solution (44) coincides with that derived in component formalism [8] through the following field redefinition $\varphi \rightarrow X, \phi$

$$\varphi \equiv e^{u+i\phi}, \quad X = |b| \tanh u, \tag{45}$$

²For simplicity, we choose m to be real positive in the following.

where u , ϕ and X are real scalar fields. After the field redefinition, the theory of massive CP^1 model is described by X and ϕ , and the wall solution (44) is mapped to

$$X = |b| \tanh |m|(y + y_0), \quad \phi = \phi_0. \quad (46)$$

This solution coincides with that derived in Ref. [8].

VI. CONCLUSION

We have constructed massive NLSMs on cotangent bundle over Grassmann manifold $T^*G_{N,M}$ and its generalization, the line bundle over $T^*G_{N,M}$ manifold in $\mathcal{N} = 1$ superfield formalism with quotient method. It was found that the former contains $N!/ [M!(N-M)!]$ vacua while the latter has no discrete vacua.

The BPS wall solution was given in $N = 2$ and $M = 1$ case of $T^*G_{N,M}$ model, which corresponds to the Eguchi-Hanson manifold. More interesting case is $N = 4$ and $M = 2$ case since it is the simplest manifold other than T^*CP^{N-1} . The theory has six discrete vacua and it is expected that the theory has various interesting wall solutions, their junction and lump.

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